Lagrangian multipliers

Lagrangian multipliers allow us to extend the idea of maximisation to more than two variables. To find the local extrema of a function $f\left(x_1,\ldots,x_n\right)$ we solve the equations:

$$\frac{\partial f}{\partial x_1}=0,\ \ \frac{\partial f}{\partial x_2}=0,\ \ \ldots,\ \ \frac{\partial f}{\partial x_n}=0$$

However the variables may be constrained in some way. We can then use the method of Lagrangian multipliers. To explain this method we will refer to this simple problem.

Example

A farmer wishes to enclose an area within a rectangular field. He wants the area enclosed to be as large as possible given the fact that he has 100m of fencing available. A straight river forms one of the boundaries of the field. What dimensions should he make the field in order to maximise the area?

In this example, we can set up the problem by letting the length and width of the rectangle be *x* and *y* respectively. Here *x* is the length of side perpendicular to the river and *y* is the length of the side parallel to the river. The problem is to maximise xy subject to the condition that $2x+y=100$.

In this very simple case it is fairly easy to see the solution:

From the constraint $y=100-2x$, we are effectively maximising $x(100-2x)$. Differentiating this we get $100-4x$, which gives $x=25$ when we set it equal to zero.

By differentiating again we can see that $x=25$ does give a maximum, so the solution is to have a field which is 25 metres by 50 metres.

However we can approach this in a different way, using Lagrangian functions. We will look at the specific case here and then generalise what you would need to do to tackle any problem.

The Lagrangian function is defined to be $L=xy-\lambda(2x+y-100)$. (Note that the constraint equation can be written as $2x+y-100=0$.)

We then solve the problem by solving $\frac{\partial L}{\partial x}=0, \frac{\partial L}{\partial y}=0, \frac{\partial L}{\partial \lambda}=0$.

> ***Question 1.1***
>
> Solve the problem using the Lagrangian function.

For more than two variables, any constraint equation can be written in the form $g(x_1,x_2,\dots,x_n)=0$, in other words, we want to solve:

Find the (local) extrema of $f(x_1,\dots,x_n)$ subject to $g(x_1,x_2,\dots,x_n)=0$

Lagrangian theory tells us to construct the Lagrangian function:

$$L=f(x_1,\dots,x_n)-\lambda g(x_1,\dots,x_n)$$

and then solve the equations:

$$\frac{\partial L}{\partial x_1}=0,\ \frac{\partial L}{\partial x_2}=0,\dots,\ \frac{\partial L}{\partial x_n}=0,\ \frac{\partial L}{\partial \lambda}=0$$

However, there may be more than one constraint. Generally, if we want to solve the problem:

Find the (local) extrema of $f(x_1,\dots,x_n)$ subject to the m constraints:

$$g_1(x_1,x_2,\dots,x_n)=0,\dots,g_m(x_1,x_2,\dots,x_n)=0$$

Lagrangian theory tells us to set up the Lagrangian function:

$$L=f(x_1,\dots,x_n)-\lambda_1 g_1(x_1,\dots,x_n)-\dots-\lambda_m g_m(x_1,\dots,x_n)$$

and then solve the equations:

$$\frac{\partial L}{\partial x_1}=0,\quad \frac{\partial L}{\partial x_2}=0,\quad \dots,\quad \frac{\partial L}{\partial x_n}=0,\quad \frac{\partial L}{\partial \lambda_1}=0,\quad \dots,\quad \frac{\partial L}{\partial \lambda_m}=0$$

> ***Question 1.2***
>
> Find the extrema of $f(x,y,z)=x^2+y^2+z^2$ subject to $2x-y+z=3$.

Solutions

Solution 1.1

$$\frac{\partial L}{\partial x} = y - 2\lambda = 0$$

$$\frac{\partial L}{\partial y} = x - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 2x + y - 100 = 0$$

Substituting the first two equations into the third, we get:

$$2\lambda + 2\lambda - 100 = 0 \quad \Leftrightarrow \quad \lambda = 25$$

This gives $x = 25$ and $y = 50$ as required.

Solution 1.2

We need to find the partial derivative equations:

$$\frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0, \frac{\partial L}{\partial z} = 0, \frac{\partial L}{\partial \lambda} = 0$$

This gives us:

$$2x - 2\lambda = 0, \quad 2y + \lambda = 0, \quad 2z - \lambda = 0, \quad 2x - y + z = 3$$

Substituting the first three equations into the fourth:

$$2\lambda + \frac{1}{2}\lambda + \frac{1}{2}\lambda = 3 \quad \Rightarrow \quad \lambda = 1$$

Hence:

$$x = 1, y = -\frac{1}{2}, z = \frac{1}{2}$$

$\lambda = 1$ doesn't mean a lot in this situation. Often this method is applied in mechanics to real physical systems. In such problems the Lagrangian multiplier, λ, does have physical meaning.